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KÄHLER MANIFOLDS WITH POSITIVE CURVATURE

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1. Introduction

This paper is concerned with the study of complex manifolds. Our main result is motivated by the following conjecture: Is a compact Kähler manifold with positive sectional curvature holomorphically equivalent to a complex projective space? The conjecture has been verified until now for complex dimension less than or equal to three, [10].

The technique used throughout this work is to consider the variational properties of the geodesic distance r of a Riemannian manifold M, where the latter is considered as a real-valued function in $M \times M$.

In § 2 we introduce the open dense submanifold of $M \times M$, denoted by $M \bigvee M$, which is the complement of the union of the diagonal submanifold of $M \times M$ and the set of cut pairs of M. The tangent bundle of $M \bigvee M$ splits into a direct sum of two subbundles V^+ and V^- both of rank dim (M). In particular, this decomposition is useful in the study of the second fundamental form of the boundary of the metric tubular neighborhoods (c-neighborhoods) of the diagonal of $M \times M$.

Since the proof of our main result requires the use of some elements on the geometry of geodesics, we include the latter in §§ 3 and 4 in the more general context of Riemannian manifolds with positive sectional curvature. Theorem 1 in § 4 gives some information about the "position" of local geodesic sprays of V^+ with respect to the metric tubular neighborhoods.

Finally, in § 5 we prove our main theorem, namely, Theorem 3: Let M be a connected compact Kähler manifold of complex dimension n with positive holomorphic bisectional curvature, then any closed n-dimensional complex analytic subvariety V (possibly singular) of $M \times M$ intersects the diagonal.

Although an alternative proof might be obtained by making use of the theory of deformations to simplify the singularities of V, we prefer a more direct method consisting of using an extended notion of the second fundamental form which applies to singular varieties.

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2. Preliminaries

Let M be a connected complete n-dimensional Riemannian manifold of class C^∞, and let us consider the product manifold $M \times M$. We distinguish in $M \times M$ the diagonal submanifold denoted by M_Δ and the set of cut pairs of M denoted by M_{cut}. The latter is a closed set of topological dimension $\leq 2n - 1$.

We consider the open submanifold

$$M \vee M = (M \times M) \backslash (M_\Delta \cup M_{cut}) .$$

The structural group of the tangent bundle of $M \vee M$, denoted by $T(M \vee M)$, can be reduced to $0(n-1)$ in a natural way as follows. Let $(p_0, q_0) \in M \vee M$, and let γ be the shortest geodesic in M between p_0 and q_0, which is unique since (p_0, q_0) is not a cut pair of M. Let us take any orthonormal frame $\{e'_i(p_0), 1 \leq i \leq n\}$ of M at p_0 such that $e'_1(p_0)$ is the unit tangent vector to γ at p_0. Then we get an orthonormal frame $\{e''_i(q_0), 1 \leq i \leq n\}$ of M at q_0 by parallel translation of the frame $\{e'_i(p_0)\}$ along γ from p_0 to q_0. After identifying $e'_i(p_0)$ with $(e'_i, 0)(p_0, q_0)$ and $e''_i(q_0)$ with $(0, e''_i)(p_0, q_0)$, we set

$$e_{\pm i}(p_0, q_0) = \frac{1}{\sqrt{2}}(e'_i(p_0) \pm e''_i(q_0)) , \qquad 1 \leq i \leq n .$$

Let us denote by $F'(M \vee M)$ the set of all frames obtained in this way. We observe that for each element in $F'(M \vee M)$ the pair (p_0, q_0) determines uniquely the vectors $e_{\pm 1}(p_0, q_0)$, and the rest of the $e_{\pm i}$'s are determined up to an element of $0(n-1)$ which acts on $F'(M \vee M)$ as follows:

$$ge_{\pm 1}(p_0, q_0) = e_{\pm 1}(p_0, q_0) ,$$
$$ge_{\pm i}(p_0, q_0) = \frac{1}{\sqrt{2}}(ge'_i(p) \pm ge''_i(q)) , \qquad i \geq 2 ,$$

for every $g \in 0(n-1)$ and $\{e_{\pm i}(p, q)\} \in F'(M \vee M)$. Accordingly the set $F'(M \vee M)$ becomes a principal bundle with $0(n-1)$ as its structural group, and is a subbundle of $F(M \vee M)$, the bundle of all orthonormal frames of $M \vee M$. The action of $0(n-1)$ as well as the product structure of $M \vee M$ as an open submanifold of $M \times M$ defines certain invariant subspaces of its tangent bundle. Among these we distinguish the following

$$V^{\pm} = \sum_{i=1}^{n} \boldsymbol{R} e_{\pm i} .$$

We shall denote by r the geodesic distance in M regarded as a real-valued function in $M \times M$. For any positive real number c, let us set

$$N_c = \{(p, q) \in M \times M / r(p, q) \leq c\} ,$$
$$W_c = \partial N_c = \{(p, q) \in M \times M / r(p, q) = c\} .$$

We call each N_c a c-neighborhood of the diagonal M_Δ in $M \times M$.

Remarks. 1. $M \vee M$ is the maximal open submanifold in $M \times M$ on which the function r is of class C^∞.

2. For each positive real number c, $W_c \cap (M \vee M)$ is a $(2n-1)$-dimensional manifold.

3. $e_{-1}(p_0, q_0)$ is the "inward" unit normal vector to W_c at (p_0, q_0), where $c = r(p_0, q_0)$.

4. Since M is complete, the product manifold $M \times M$ is also complete, so that the exponential map of $M \times M$ is defined on the whole $T(M \times M)$. Let (p_0, q_0) be an element in $M \vee M$, and $c = r(p_0, q_0)$. Then the connected component of (p_0, q_0) in the intersection of $\bigcup_{t \in R} \exp_{(p_0, q_0)} (te_{+1})$ with $M \vee M$ is contained in W_c.

5. e_{+1} is a principal direction of curvature in the tangent space of W_c, and its corresponding principal curvature is equal to zero. This follows from the fact that

$$\nabla_{e_{+1}} e_{-1} = 0 ,$$

where ∇ stands for the covariant differentiation in the Levi-Civita connection of $M \times M$.

3. Riemannian manifolds with positive sectional curvature

In the present and next sections we obtain some information about the boundary of the c-neighborhoods of M as well as the "position" of n-dimensional local geodesic sprays in $M \times M$ with respect to the c-neighborhoods by assuming that the sectional curvature of M is positive. First of all, we prove

Proposition 1. *Let $(p_0, q_0) \in M \vee M$ and $c = r(p_0, q_0)$. Then W_c has at (p_0, q_0) at least $n-1$ principal directions, in which the normal curvature is positive (i.e., W_c is at least $(n-1)$-concave at (p_0, q_0)), and at least one principal curvature equal to zero.*

Proof. The proof makes use of the variation of the length integral of a one-parameter family of curves to show that the second fundamental form of W_c is positive semi-definite on V^+. Let γ be the shortest geodesic in M between p_0 and q_0. Then we may assume that γ is parametrized by the arc-length s, $0 \leq s \leq c$. For each v in $V^+_{(p_0, q_0)} \backslash \boldsymbol{R} e_{+1}(p_0, q_0)$ we construct a one-parameter family of curves $\gamma_t(s), 0 \leq s \leq c$ and $|t| < \varepsilon$ for some positive number ε, having the foollowing properties:

(i) $\gamma_0(s) = \gamma(s)$, for all $s \in [0, c]$.

(ii) $\gamma_t(0) = \exp_{p_0}(tv') = p_t$ and $\gamma_t(c) = \exp_{q_0}(tv'')$ for all $|t| < \varepsilon$, where $v' \in T_{p_0}(M) \backslash \boldsymbol{R} e'_1$, $v'' \in T_{q_0}(M) \backslash \boldsymbol{R} e''_1$ and $v = v' + v''$.

(iii) For every $s \in (0, c)$, $\gamma_t(s)$ is the geodesic tangent to $v'(s)$, where $v'(s)$ is the parallel translate of v' along γ from p_0 to $\gamma(s)$.

Let us denote by L the length integral of a curve. Then

$$(1) \qquad L(\gamma_t) \geq r(\exp_{(p_0,q_0)}(tv)) = r(p_t, q_t)$$

for every $|t| < \varepsilon$. By the construction of $\gamma_t(s)$, we have

$$L(\gamma_0) = r(p_0, q_0) , \qquad \frac{d}{dt}L(\gamma_t)|_{t=0} = 0 ,$$

and also

$$\frac{d}{dt}r(p_t, q_t)|_{t=0} = dr_{(p_0,q_0)}(v) = 0 .$$

Then from (1) we get

$$(2) \qquad \frac{d^2}{dt^2}L(\gamma_t)|_{t=0} \geq \frac{d^2}{dt^2}r(p_t, q_t)|_{t=0} .$$

Next, by computing the second variation of the arc length with respect to the family $\gamma_t(s)$ we obtain ([2], [6])

$$(3) \qquad \frac{d^2}{dt^2}L(\gamma_t)|_{t=0} < 0 .$$

On the other hand

$$(4) \qquad \frac{d^2}{dt^2}r(p_t, q_t)|_{t=0} = r_{,ij}v^iv^j = -\Pi_{W_c}[v] ,$$

where $r_{,ij}$ and Π_{W_c} stand for the second covariant differentiation in the Levi-Civita connection of M (in a coordinate system of M at (p_0, q_0)) and the second fundamental form of W_c at (p_0, q_0) respectively, and the repetition of indices indicates summation.

From (3), (4) and (2) we conclude

$$\Pi_{W_c}[v] > 0 .$$

This together with Remark 5 shows that Π_{W_c} is positive semi-definite on $V^{+}_{(p_0,q_0)}$. Hence W_c must have at least $n-1$ positive principal directions of curvature and at least one equal to zero (the e_{+1}) at (p_0, q_0) proving our assertion.

Now let $(p_0, q_0) \in M \vee M$, and $c = r(p_0, q_0)$. We shall prove that there exists a neighborhood U_ε of 0 in $V^{+}_{(p_0,q_0)}$ such that

$$r(\exp_{(p_0,q_0)}(x)) \leq r(p_0, q_0)\ , \qquad \text{for all } x \in U_\varepsilon\ .$$

Let S be a local geodesic spray of V^{+} at (p_0, q_0), and let $(u_1, \cdots, u_n)$ be a coordinate system of S with center at (p_0, q_0) such that

$$u_i \left(\exp_{(p_0,q_0)} \left(\sum_{j=1}^{n} t_j e_{+j}\right)\right) = t_i\ , \qquad 1 \leq i \leq n\ ,$$

where we have chosen the e_{+i}'s, $2 \leq i \leq n$, to be the principal directions of curvature of W_c at (p, q) with principal curvatures $\mathscr{K}_i\ (> 0)$, $2 \leq i \leq n$ (see Proposition 1).

We proceed to show that for every $(p_s, p_s) \in S$ with coordinates

$$u_1((p_s, q_s)) = s\ ,$$
$$u_i((p_s, q_s)) = 0\ , \qquad 2 \leq i \leq n\ ,$$

the differential of r satisfies

$$dr_{(p_s,q_s)}(v) = 0 \tag{5}$$

for every $v \in T_{(p_s,q_s)}(S)$.

To prove (5) it will be sufficient to prove

$$T_{(p_s,q_s)}(S) \subseteq (T_{(p_s,q_s)}(M \vee M)) \vdash e_{-1}\ , \tag{6}$$

where $\vdash$ stands for perpendicular in the natural metric of $M \times M$. Let $v \in T_{(p_s,q_s)}(S)$. The knowledge of one of the projections of v either onto $T_{p_s}(M)$ or onto $T_{q_s}(M)$ determines the other. In fact, let us assume given v' the projection of v onto $T_{p_s}(M)$ and let us determine v'' in $T_{q_s}(M)$ such that $v' + v'' = v$. Let α be a curve in M through p_s tangent to v', i.e.,

$$\alpha(t) = \exp_{p_0}(v'(t))\ ,$$

with $v'(t) \in T_{p_0}(M)$ and $|t| < \eta$ for some positive real number η. Next, by parallel translation of $v'(t)$ along γ from p_0 to q_0 we obtain $v''(t) \in T_{q_0}(M)$, and then

$$\beta(t) = \exp_{q_0}(v''(t))\ ,$$

with $|t| < \eta$, is a curve in M through q_s whose tangent v'' belongs to $T_{q_s}(M)$ and

$$v' + v'' = v\ .$$

Remarks. (i) If v' is tangent to γ at p_s, then our construction shows that v'' is tangent to γ at q_s, and also that $v = v' + v''$ belongs to $\boldsymbol{R}e_{+1}$.

(ii) If v' belongs to $(T_{p_s}(M)) \vdash e'_1(p_s)$, then our argument together with the Gauss lemma, [2], shows that v'' belongs to $(T_{q_s}(M)) \vdash e_1''(q_s)$, and therefore $v = v' + v''$ belongs to $(T_{(p_s,q_s)}(M \vee M)) \vdash Re_1$.

Remarks (i) and (ii) prove (6) and hence (5).

In the coordinates $(u_1, \cdots, u_n)$ the function r can be expressed as

$$r(u_1, \cdots, u_n) = r(p_0, q_0) - \frac{1}{2} \sum_{i=2}^{n} \mathscr{K}_i u_i^2 + \sum_{i,j=2}^{n} u_i u_j \varphi_{ij}(u) ,$$

where each $\varphi_{ij}\ (u_1, \cdots, u_n)$, $2 \leq i, j \leq n$, is at least linear in u_1 because of (5). Therefore we can restrict the u_i's conveniently so that

$$r(p, q) \leq r(p_0, q_0) ,$$

for every (p, q) in S, proving our assertion.

4. Cut pairs

In this section by dealing with cut pairs of M we obtain a refinement (Theorem 1) of the result proved at the end of the last section. Let $(p_0, q_0) \in W_c$, $c = r(p_0, q_0)$, and let us assume that $(p_0, q_0) \in M_{cut}$. Take a shortest geodesic γ in M between p_0 to q_0, and let us consider m_0 to be any point on γ different from p_0 and q_0. Then neither (p_0, m_0) nor (m_0, q_0) is a cut pair, and therefore we can apply the result at the end of § 3 to get

$$r(p, m) \leq r(p_0, m_0) , \tag{7}$$

$$r(m, q) \leq r(m_0, q_0) \tag{8}$$

for every (p, m) and (m, q) belonging to the local geodesic sprays of V^+ through (p_0, m_0) and (m_0, q_0) respectively.

Next, by using the inequalities (7) and (8) and the fact that γ is a geodesic, we get

$$r(p, q) \leq r(p, m) + r(m, q) \leq r(p_0, m_0) + r(m_0, q_0) = r(p_0, q_0)$$

for every (p, q) in a local geodesic spray of V^+ (constructed from r) at (p_0, q_0). Finally, we can state

Theorem 1. *Let M be a connected complete n-dimensional Riemannian manifold with positive sectional curvature. Then for each (p_0, q_0) in $M \times M$ there exists a neighborhood U_ε of 0 in $V^+_{(p_0,q_0)}$ such that*

(i) $\exp_{(p_0,q_0)}(x) \in N_c$, *for every $x \in U_\varepsilon$* ,

(ii) $\dim((\exp_{(p_0,q_0)}(U_\varepsilon)) \cap W_c) \leq 1$,

where $c = r(p_0, q_0)$.

As an application we shall prove

Theorem 2. *Let M be as in Theorem 1.*

(a) *Let V^n be an n-dimensional local geodesic spray at (p_0, q_0), and assume that r attains its minimum on V^n at (p_0, q_0). Then the set C of all points in V^n*

where r is equal to c ($= r(p_0, q_0)$) includes at least one geodesic in which the point (p_0, q_0) is an interior point.

(b) *Let V^n be an n-dimensional submanifold of $M \times M$ transversal to e_{+1}. Then r cannot achieve its minimum on V^n at a point (p_0, q_0) which is flat relative to e_{-1} (i.e., the e_{-1} component of the second fundamental form of V^n at (p_0, q_0) is zero).*

Proof. (a$_1$) Let us assume that (p_0, q_0) is not a cut pair. Then we have

$$T_{(p_0,q_0)}(V^n) \subseteq T_{(p_0,q_0)}(W_c) ,$$

since r has a minimum on V^n at (p_0, q_0). Moreover,

$$V^+_{(p_0,q_0)} \subseteq T_{(p_0,q_0)}(W_c) ,$$
$$\dim (T_{(p_0,q_0)}(V^n)) = \dim (V^+_{(p_0,q_0)}) = n ,$$

and therefore

$$T_{(p_0,q_0)}(V^n) \cap V^+_{(p_0,q_0)}) \neq (0) .$$

Let $v_0 \in (T_{(p_0,q_0)}(V^n) \cap V^+_{(p_0,q_0)}) \backslash \{0\}$. Then by Proposition 1

$$\Pi_{W_c}[v_0] \geq 0 . \tag{9}$$

On the other hand

$$\Pi_{W_c}|_{T_{(p_0,q_0)}(V^n)} \geq 0 . \tag{10}$$

From (9) and (10), we have

$$\Pi_{W_c}[v_0] = 0 , \tag{11}$$

and from (11),

$$T_{(p_0,q_0)}(V^n) \cap V^+_{(p_0,q_0)} = \boldsymbol{R}e_{+1} .$$

Since V^n is a local geodesic spray at (p_0, q_0), $\exp_{(p_0,q_0)} (te_{+1})$ belongs to V^n for every $t \in (a, b)$, where $a < 0$ and $b > 0$.

Set

$$C = \{(p, q) \in V^n / r(p, q) = r(p_0, q_0) = c\} .$$

It is clear that $\exp_{(p_0,q_0)} (te_{+1})$ is contained in C for every t, $a < t < b$, since (p_0, q_0) is not a cut pair.

(a$_2$) Now let us assume that $(p_0, q_0) \in M_{cut}$. In this case we shall introduce an auxiliar function r' which is of class C^∞ in a neighborhood of (p_0, q_0) and satisfies

(12) $$r'(p_0, q_0) = r(p_0, q_0) ,$$

(13) $$r'(p, q) \geq r(p, q) ,$$

wherever it is meaningfull.

Definition of r'. Let us take γ to be a shortest geodesic in M between p_0 and q_0, and let m_0 be any point on γ different from p_0 and q_0. Then we set

$$r'(p, q) = r'(\exp_{p_0}(x), \exp_{q_0}(y)) = r(p, m(p, q)) + r(m(p, q), q) ,$$

where $(p, q) = \exp_{(p_0, q_0)}(x, y), x \in T_{p_0}(M), y \in T_{q_0}(M)$, and

$$m(p, q) = \exp_{m_0} \frac{1}{c}[(c - s_0)x' + s_0 y'] ,$$

where x' and y' are the parallel translations along γ of x and y from p_0 to m_0 and q_0 to m_0 respectively, and $s_0 = r(p_0, m_0)$.

The r' just defined has the required properties (12) and (13), so finally by setting

$$N' = \{(p, q) / r'(p, q) \leq r(p_0, q_0)\} ,$$
$$W' = \{(p, q) / r'(p, q) = r(p_0, q_0)\} ,$$

we have that W' is a (2n — 1)-dimensional manifold and observe that if r attains a minimum at (p_0, q_0) it also holds for r'. Therefore the case of the cut pair (p_0, q_0) will be reduced to the non-cut pair case (a_1) by replacing r by r'.

(b) Let us assume that r achieves its minimum on V^n at (p_0, q_0) which is a flat point relative to e_{-1}. Then we get

$$T_{(p_0, q_0)}(V^n) \cap (V^+_{(p_0, q_0)} \backslash \boldsymbol{R}e_{+1}) \neq (0)$$

because of the minimality of r at (p_0, q_0) and the transversality of V^n with respect to e_{+1}.

Let $v_0 \in (T_{(p_0, q_0)}(V^n) \cap (V^+_{(p_0, q_0)} \backslash \boldsymbol{R}e_{+1})) \backslash \{0\}$. Then

(14) $$\Pi_{W_c}[v_0] > 0$$

from Proposition 1. On the other hand

(15) $$\Pi_{W_c}|_{T_{(p_0, q_0)}(V^n)} \leq 0$$

because of the minimality of r at (p_0, q_0).

The inequalities (14) and (15) lead us to a contradiction, hence r cannot achieve its minimum on V^n at a flat point with respect to e_{-1}. This concludes the proof of (b) and hence that of Theorem 2.

5. Complex manifolds

Let M be a complex manifold of complex dimension n, and let $(U; z_1, \cdots, z_n)$ be a coordinate system of M at $p \in M$ with origin at p. We proceed to define a quadratic transformation at the point p ("blowing-up"). Let $p^{n-1}(C)$ be an $(n-1)$-dimensional complex projective space with homogeneous coordinates $w_1, \cdots, w_n$, and consider the complex manifold $U \times p^{n-1}(C)$.

Let $\hat{U} \subseteq U \times p^{n-1}(C)$ be defined by

$$\hat{U} = \{(q, \zeta) \in U \times p^{n-1}(C) / z_i(q) w_j(\zeta) = z_j(q) w_i(\zeta),\ 1 \leq i, j \leq n\} .$$

U is a complex manifold of complex dimension n. In fact, if we set V_k equal to the subset of Cp^{n-1} where $w_k \neq 0$, then

$$p^{n-1}(C) = \bigcup_{k=1}^{n} V_k ,$$

and in $(U \times V_k) \cap \hat{U}$ the defining equations give $z_j = z_k w_j / w_k$ so that $(z_k, w_1/w_k, \cdots, w_{k-1}/w_k, \cdots, w_n/w_k)$ forms a coordinate system in $(U \times V_k) \cap \hat{U}$.

We define a projection $\sigma_U : \hat{U} \to U$ by $\sigma_U(q, \zeta) = q$, which is one-to-one except in $\sigma_U^{-1}(\{p\})$ because if $q \neq p$ there exists j, $1 \leq j \leq n$, such that $z_j(q) \neq 0$, then $w_k = w_j z_k / z_j$. Hence the w's are determined up to a proportionality implying the existence of a unique $\zeta \in p^{n-1}(C)$ such that $\sigma_U(q, \zeta) = q$.

Let us denote by σ_1 the restriction of σ_U to $\hat{U} \backslash \sigma_U^{-1}(\{p\})$. Then we can define a complex manifold by setting

$$\hat{M}_p = \hat{U} \bigcup_{\sigma_1} (M \backslash \{p\}) ,$$

where the symbol $\bigcup_{\sigma_1}$ denotes the union of $\hat{U}$ with $M \backslash \{p\}$ in which the respective subsets $\hat{U} \backslash \sigma_U^{-1}(\{p\})$ and $U \backslash \{p\}$ are identified under σ_1. One gets a manifold from the fact that the graph of σ_1 in $\hat{U} \times M \backslash \{p\}$ is a closed subspace, [3].

There is a natural map $\sigma : \hat{M}_p \to M$, which extends the σ_U and is also onto and one-to-one except in $\sigma^{-1}(\{p\})$. The subvariety $\sigma^{-1}(\{p\})$ is an $(n-1)$-dimensional complex projective space and will be denoted by B_p. The manifold $\hat{M}_p$ is called the "blowing-up" manifold of M at the point p, and σ a quadratic transformation with respect to p. For any two coordinate systems $(U; z_1, \cdots, z_n)$ and $(U'; z_1', \cdots, z_n')$ in neighborhoods of p with origin at p, the natural isomorphism of $\hat{M}_p \backslash \sigma^{-1}(\{p\})$ and $M_p' \backslash \sigma'^{-1}(\{p\})$ extends naturally to a holomorphic isomorphism, [1] and [4].

We consider now the effect of the "blowing-up" of M at p on a subvariety V containing p.

Lemma 1. *Let $V \subseteq M$ be an analytic subvariety and let $p \in V$, and consider $\hat{M}_p$ with the subvariety $V_p^0 = \sigma^{-1}(V \backslash (\{p\}))$. Then the topological closure $\hat{V}_p$ of V_p^0 in $\hat{M}_p$ is an analytic subvariety of $\hat{M}_p$.*

Proof. We consider $\sigma^{-1}(V)$, which is an analytic subvariety of $\hat{M}_p$ and is a finite union of irreducible components, [8]; let us say

$$\sigma^{-1}(V) = B_p \cup A_1 \cup \cdots \cup A_m ,$$

where each A_i, $1 \leq i \leq m$, is an irreducible component. Denote by A the analytic subvariety $A_1 \cup \cdots \cup A_m$, and observe that A must contain the set V_p^0. Since σ is one-to-one in the complement of B_p, we get

$$A \backslash A \cap B_p = V_p^0 . \tag{16}$$

Next, by taking closure in $\hat{M}_p$, the identity (16) becomes

$$A = \overline{A \backslash A \cap B_p} = \overline{V_p^0} = \hat{V}_p , \tag{17}$$

since $A \backslash A \cap B_p$ is everywhere dense in A. The identity (17) proves the lemma.

The subvariety $\hat{V}_p$ is called the "blowing-up" of the variety V at the point p. Denote it by $K_p(V) = \hat{V}_p \backslash V_p^0 = \hat{V}_p \cap B_p$ and call it the *projective tangent cone* of the variety V at p. Note that if V is irreducible and d-dimensional, then the dimensions of $\hat{V}_p$ and $K_p(V)$ are d and $d-1$ respectively.

Now let us consider M to be a Kähler manifold, and let us denote by R and J its Riemann curvature tensor and the automorphism of $T(M)$ with $J^2 = -\text{id.}$, induced by the complex structure of M, respectively.

Definition. Let M be a Kähler manifold, and let σ and σ' be two J-invariant planes in $T_p(M)$. Then the *holomorphic bisectional curvature* $H(\sigma, \sigma')$ is defined [7] by

$$H(\sigma, \sigma') = R(t, Jt, s, Js) ,$$

where t and s are unit vectors in σ and σ' respectively. By using Bianchi's identity we have

$$H(\sigma, \sigma') = R(t, s, t, s) + R(t, Js, t, Js) .$$

Finally, by taking under consideration Kähler manifolds with positive holomorphic bisectional curvature, we prove as the main result in this paper the following generalization of a result in [6].

Theorem 3. *Let M be a compact connected Kähler manifold of complex dimension n with positive holomorphic bisectional curvature. Then any closed n-dimensional complex analytic (possibly singular) subvariety V of $M \times M$ intersects M_Δ.*

Proof. We shall reach a contradiction by assuming that r achieves a positive relative minimum on V at (p_0, q_0). Since the case of the cut pair (p_0, q_0) can be reduced to the noncut pair case by introducing an auxiliary function r' (Theorem 2, § 4), we are just left with the following two cases.

(i) Let (p_0, q_0) be an element in $M \vee M$, and assume that it is a regular point of V. In this case, we have

$$T_{(p_0,q_0)}(V) \subseteq T_{(p_0,q_0)}(W_c) ,$$
$$T_{(p_0,q_0)}(W_c) = \sum_{i=1}^{n} \boldsymbol{R}e_i + \sum_{i=2}^{n} \boldsymbol{R}e_{-i} + \sum_{i=1}^{n} \boldsymbol{R}Je_i + \sum_{i=1}^{n} \boldsymbol{R}Je_{-i} ,$$

where $c = r(p_0, q_0)$, and J is the automorphism of $T_{(p_0,q_0)}(M \times M)$ with $J^2 = -\text{id.}$, defined by the complex structure of $M \times M$.

Let us set

$$V^{+}_{(p_0,q_0)} = \sum_{i=1}^{n} \boldsymbol{R}e_i = \sum_{i=1}^{n} \boldsymbol{R}Je_i .$$

Then we have

$$\dim_{\boldsymbol{R}} (T_{(p_0,q_0)}(W_c)) = 4n - 1 ,$$
$$\dim_{\boldsymbol{R}} (V^{+}_{(p_0,q_0)}) = \dim_{\boldsymbol{R}} (T_{(p_0,q_0)}(V)) = 2n .$$

Hence

$$\dim_{\boldsymbol{R}} ((T_{(p_0,q_0)}(V)) \cap V^{+}_{(p_0,q_0)}) \geq 1 .$$

Let $v_0 \in (T_{(p_0,q_0)}(V) \cap V^{+}_{(p_0,q_0)})\backslash\{0\}$. Then Jv_0 belongs to $(T_{(p_0,q_0)}(V) \cap V^{+}_{(p_0,q_0)})\backslash\{0\}$, and v_0 and Jv_0 are $\boldsymbol{R}$-linearly independent. Therefore

$$\dim_{\boldsymbol{R}} (T_{(p_0,q_0)}(V) \cap V^{+}_{(p_0,q_0)}) \geq 2 .$$

Now we make use of the following relations:

$$\Pi_{W_c}|_{T_{(p_0,q_0)}(V)} \leq \Pi_V , \tag{18}$$

$$\Pi_V[v] + \Pi_V[Jv] = 0 , \tag{19}$$

for all $v \in T_{(p_0,q_0)}(V)$, where Π_V stands for the component of the second fundamental form of V in the direction of the "outward" normal to W_c.

Let $v \in T_{(p_0,q_0)}(V)\backslash \boldsymbol{R}e_1 \cup \boldsymbol{R}Je_1$. Then we have

$$0 < \Pi_{W_c}[v] + \Pi_{W_c}[Jv] = -2r_{,\alpha\bar{\beta}}v^{\alpha}v^{\bar{\beta}} \tag{20}$$

by using the fact that M is a Kähler manifold with positive holomorphic bisectional curvature and a computation similar to that carried out in the proof of Proposition 1 in § 2. Therefore

$$\Pi_V[v_0] + \Pi_V[Jv_0] > 0 \tag{21}$$

for all v_0 in $V^{+}_{(p_0,q_0)} \cap T_{(p_0,q_0)}(V)\backslash \boldsymbol{R}e_1 \cup \boldsymbol{R}Je_1$, because of (20) and (18).

On the other hand

$$\Pi_V[v_0] + \Pi_V(Jv_0] = 0 \tag{22}$$

because of (19). Subtracting (21) from (22) we have

$$(\Pi_V - \Pi_{W_c})[v_0] + (\Pi_V - \Pi_{W_c})[Jv_0] < 0 ,$$

leading to a contradiction, since each summand is positive or zero because of (18).

(ii) Let (p_0, q_0) be an element of $M \vee M$, and assume that it is a singularity of V. In this case, we proceed as follows. Let us consider submanifolds M_1 and M_2 of $M \vee M$ containing (p_0, q_0) with

$$T_{(p_0,q_0)}(M_1) = T_{(p_0,q_0)}(W_c) \cap J(T_{(p_0,q_0)}(W_c)) ,$$
$$T_{(p_0,q_0)}(M_2) = V^+_{(p_0,q_0)} ,$$

respectively. Hence

$$K_{(p_0,q_0)}(M_2) \subseteq K_{(p_0,q_0)}(M_1) .$$

On the other hand

$$K_{(p_0,q_0)}(V) \subseteq K_{(p_0,q_0)}(M_1)$$

because of the minimality of the function r at (p_0, q_0). Therefore

$$K_{(p_0,q_0)}(V) \cap K_{(p_0,q_0)}(M_2) \neq \emptyset ,$$

since $K_{(p_0,q_0)}(V)$ is an algebraic variety by Chow's theorem [5], and the dimensions of $K_{(p_0,q_0)}(V)$ and $K_{(p_0,q_0)}(M_2)$ are complementary dimensional in $K_{(p_0,q_0)}(M_1)$ ($=(2n-2)$-dimensional complex projective space).

Let A_0 be an element in $K_{(p_0,q_0)}(V) \cap K_{(p_0,q_0)}(M_2)$. We may assume also that it belongs to neither $\boldsymbol{R}e_1$ nor $\boldsymbol{R}Je_1$. Then by Lemma 1 there exists a holomorphic curve $\varphi(t)$ in $\hat{V}_p$ such that $\varphi(0) = A_0$ and with the property that it intersects B_p just at A_0 locally. The projection of $\varphi(t)$ under σ provides us with a holomorphic curve $C(t)$ in V, with $C(0) = (p_0, q_0)$ and such that if $(z_1, \cdots, z_n)$ is a coordinate system in a neighborhood of (p_0, q_0) with origin at (p_0, q_0), there exists an integer $d \geq 1$ such that

$$z_\alpha(C(t)) = (C(t))^\alpha = \sum_{j=0}^{\infty} A_j^\alpha t^{d+1}$$

with $A_0^\alpha \neq 0$ for some α, $1 \leq \alpha \leq 2n$, since the A_0's are the local homogeneous coordinates of the point A_0 in $\hat{M}_p$.

We are going to show that for sufficiently small nonzero t

$$(23) \qquad r(C(t)) - r(C(0)) < 0 \, .$$

We recall that the jets of a differentiable real-valued function on C^m have bidegree (d', d'') and real (or total) degree $d = d' + d''$.

In the coordinate system $(z_1, \cdots, z_{2n})$ we can write

$$\begin{aligned} r(C(t)) - r(C(0)) = 2\mathscr{R}e(r,_{\alpha}(C(t))^{\alpha}) &+ \mathscr{R}e(r,_{\alpha\beta}(C(t))^{\alpha}(C(t))^{\beta}) \\ &+ r,_{\alpha\bar\beta}(C(t))^{\alpha}(C(t))^{\bar\beta} + \text{terms in } (t, \bar t) \text{ of} \\ &\quad \text{total degree greater than or equal to } 3d \, , \end{aligned}$$

where the barred indices $\bar\alpha = \alpha + 2n, \cdots$ range from $2n + 1$ to $4n$ and refer to the conjugate holomorphic coordinates $z_{\alpha+n} = z_{\bar\alpha} = \bar z_{\alpha}$.

Next, we take the average of the function $r(C(t)) - r(C(0))$, i.e.,

$$\frac{1}{2\Pi} \int_0^{2\Pi} (r(C(te^{i\theta})) - r(C(0))) d\theta \, .$$

On the other hand

$$\begin{aligned} & r(C(te^{i\theta})) - r(C(0)) \\ &= 2\mathscr{R}e\left(r,_{\alpha}\left(\sum_{j=0}^{\infty} A_j^{\alpha} t^{d+j} e^{i\theta(d+j)}\right)\right. \\ &\quad + \mathscr{R}e\left[r,_{\alpha\beta}\left(\sum_{j=0}^{\infty} A_j^{\alpha} t^{d+j} e^{i\theta(d+j)}\right)\left(\sum_{k=0}^{\infty} A_k^{\beta} t^{d+k} e^{i\theta(d+k)}\right)\right] \\ &\quad + r,_{\alpha\bar\beta}\left[\left(\sum_{j=0}^{\infty} A_j^{\alpha} t^{d+j} e^{i\theta(d+j)}\right)\left(\sum_{k=0}^{\infty} A_k^{\bar\beta} t^{d+k} e^{-i\theta(d+k)}\right)\right] \\ &\quad + \text{terms in } (t, \bar t) \text{ of total degree greater than or equal to } 3d \, . \end{aligned}$$

We observe that the only summand in the above expression giving any contribution when integrated from 0 to 2Π comes from

$$r,_{\alpha\bar\beta}\left[\left(\sum_{j=0}^{\infty} A_j^{\alpha} t^{d+j} e^{i\theta(d+j)}\right)\left(\sum_{k=0}^{\infty} A_k^{\bar\beta} t^{d+k} e^{-i\theta(d+k)}\right)\right]$$

and is given by the jet of total degree less that $3d$ of

$$r,_{\alpha\bar\beta}\left(\sum_{j=k=0}^{\infty} A_j^{\alpha} A_k^{\bar\beta} t^{d+j} \bar t^{d+k}\right) .$$

Therefore

$$\begin{aligned} & \frac{1}{2\Pi} \int_0^{2\Pi} (r(C(te^{i\theta})) - r(C(0))) d\theta \\ & \qquad = \frac{1}{2\Pi} \int_0^{2\Pi} r,_{\alpha\bar\beta}(A_0^{\alpha} A_0^{\bar\beta} \, |t|^{2a} + A_1^{\alpha} A_1^{\bar\beta} \, |t|^{2(d+1)} + \cdots) d\theta \end{aligned}$$

$$= r,_{\alpha\beta}\left(\sum_{j=0}^{\infty} A_j^\alpha A_j^\beta \; |t|^{2(d+j)} \right) ,$$

which is negative for small t because of (20). This proves the inequality (23) which contradicts the fact that r achieves a minimum on V at (p_0, q_0). Therefore we conclude that r cannot achieve a positive minimum on V. On the other hand, the compactness of V in $M \times M$ and the continuity of r imply the existence of some (p, q) in V, which achieves a minimum on V, and by our discussion $r(p, q)$ must be equal to zero, which shows the case (ii). Hence the proof of Theorem 3 is complete.

Ih the case where V has no singularities our Theorem 3 is [6] equivalent to Theorem 2, which was used to prove that a compact Kähler manifold of complex dimension 2 and positive sectional curvature is analytically isomorphic to $P_2(C)$, but so far our technique does not seem to be applicable to study the conjecture for greater dimensions.

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